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On the mixing of angular momentum in a stirred rotating fluid

By F. P. BRETHERTON AND J. S. TURNER
Department of Applied Mathematics and Theoretical Physics,
University of Cambridge, England

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It has been suggested on theoretical grounds that a vortex could be initiated in a cylindrical region of fluid, originally in solid rotation, by the horizontal mixing of angular momentum produced by external stirring. In this paper various arguments for and against the mechanism are examined and their difficulties exposed. No firm conclusion is reached. A series of mathematical models of the mixing motions has been used, to bring out the differences between mechanical stirring and the agitation of a gas by random molecular motions. They suggest the introduction of a diffusion coefficient for angular momentum, to be determined empirically. These theoretical ideas are then applied to the interpretation of the results of a laboratory experiment which has been designed to test the proposed mechanism directly.

A wide, flat tank of liquid was set up on a rotating table and stirred with a vertically oscillated grid, whose elements were much smaller than the width of the tank. A neutrally buoyant particle was used as a tracer of fluid motions, to measure relative circulation velocities and the properties of the turbulence. The motion observed was dominated by the loss of angular momentum to the walls and the grid, an effect which has not been taken into account in previous theoretical assessments of the effects of mixing of angular momentum. The relative circulation present was not significantly different from zero, and the limits of error of the measurements imply that the rate of diffusion of angular momentum is less than 5 % of that for fluid particles, with 95 % probability.

1. Introduction

Scorer (1965, 1966) has suggested that, when a cylindrical region of fluid is rotating about its axis and is stirred on a small scale by an external agency, a single large-scale vortex will tend to form in which the mean swirl velocity $\bar{v}(r)$ relative to an inertial frame varies inversely as the distance r from the axis of rotation; i.e. the mean circulation is

$$\bar{\gamma} = r\bar{v} = \text{constant}. \tag{1}$$

This is quite different from the equilibrium state of motion of an unstirred fluid,

$$\bar{\gamma} = \Omega r^2, \tag{2}$$

where Ω = constant. This suggestion has given rise to considerable unpublished discussion, particularly at the *Symposium on Concentrated Vortices* at Ann

Arbour in July 1964 (Küchemann 1965). It is not easy to make quantitative the theoretical arguments that have been put forward, by Scorer and others, in favour of a stirred fluid tending towards the state (1), rather than state (2), but in view of the far-reaching nature of the conclusion, and the apparently favourable experiments of Gough & Lynden-Bell (1968), a more precise investigation of possible mechanisms seems desirable.

A superficially attractive argument is that the angular momentum per unit mass of a fluid particle about the axis of rotation is $\gamma = vr$. The total angular momentum of any set of interacting fluid particles is a conserved quantity, and any conservative quantity is *a priori* uniform in a perfectly mixed state. The angular momentum is uniform in state (1) but not in state (2). Hence, given adequate externally-driven stirring, state (1) should be preferred. In a uniformly rotating fluid a particle crossing the cylindrical surface of radius r travelling inwards comes from a region where the mean angular momentum $\overline{\gamma}$ is larger, and fluctuations γ' about the mean are symmetrically distributed. Such particles apparently carry with them on the average an excess of angular momentum over the mean value at r, and similarly outward moving particles carry a deficiency. The net result is apparently a down-gradient flux of angular momentum. This down-gradient flux has been expressed quantitatively in terms of a mixing length for the turbulence by Prandtl (1931). It tends to make $\overline{\gamma}$ uniform, and vanishes only when it is so.

One immediate puzzle about this reasoning arises from its generality. Why should the same conclusion not hold for the particles of a gas stirred by random thermal motions of the molecules (the reservoir of energy associated with these is much larger than that required to build up a vortex at low Mach number)? The second law of thermodynamics assures us that it cannot, but nevertheless it is important to appreciate exactly where the argument fails, so that the same fallacy should not be transferred to the more difficult case of a turbulent liquid. This is discussed in §2(d). Another obvious question is how the axis is located dynamically. In an incompressible homogeneous liquid with rigid boundaries, the only influence of a uniform rotation is through the Coriolis force. If the turbulence is also isotropic there is no preferred direction locally, and there is no obvious reason why the angular momentum which is mixed should be about the axis of rotation rather than about any other axis.

Another argument relies on the familiar restoring force associated with infinitesimal displacements from an axisymmetric state of motion in which $\overline{\gamma}(r)$ increases with r. This force was first described by Rayleigh (1916), who showed Couette flows to be stable to axisymmetric inviscid disturbances. It also underlies non-axisymmetric Rossby waves in the atmosphere. The force may vanish for certain special classes of displacement (e.g. two-dimensional ones in a uniformly rotating flow), but when it is non-zero it always tends to restore a particle to its original radius. If, the argument runs, a fluid with such a mean circulation is stirred by an external agency which exerts no torque, work must be done by the external agency against this restoring force, and this work must appear as an increase in kinetic energy of the mean swirl. This should then increase until the restoring force vanishes, i.e. until the circulation $2\pi\overline{\gamma}(r)$ is independent of radius.

A similar approach was used by Prandtl (1931) to estimate the energy available to drive turbulence in a swirling flow.

It will be seen below that neither of these arguments can be accepted without major modification; their deceptive simplicity conceals substantial difficulties. However, neither can the possibility they raise be definitely ruled out.

In their pure form, the above arguments assume that there is no transfer of angular momentum from the container walls to the fluid, and they are not applicable in the region very near the axis in which the circulation and angular momentum must increase rapidly with radius and mixing is inhibited. Also it should be emphasized that the stirring envisaged here is on a scale small compared to the size of the container, and is not directly driving a mean meridional circulation. If there is an outward mean radial motion at one level, the fluid will tend to swirl less rapidly as it moves away from the axis of rotation, and it will have an anticyclonic circulation relative to the rotating container, whereas associated with inflow at another level will be a cyclonic vortex. If the outflow takes place near a rigidly rotating wall, and the inflow near a free surface, the anticyclonic circulation will be suppressed, and the vertically averaged velocity $\bar{v}$ will be larger than that for solid body rotation.

When these ideas first came to our attention, some preliminary observations were made in a cylindrical container 22 cm in diameter and about 20 cm deep. This was put on a turn-table and set into solid rotation at 60 r.p.m. and then stirred by a double pass of a plane circular grid with a square mesh of 1·5 mm thick plexiglass strips 1 cm wide and spaced at 5 cm centres. The grid was not rotating relative to the laboratory and no rotoscope was available to view the motion relative to the rotating frame. As the turbulence induced by the stirring died away, marker particles both in suspension and floating on the surface indicated the appearance of one, or sometimes two, cyclonic vortices roughly along the axis of the cylinder. The interpretation of these observations is ambiguous. In particular, it is not clear how much of the effect was due to a meridional circulation, or whether the observed vortex was an amalgamation of the small number of individual vortices associated with a less conspicuous region of more slowly rotating fluid elsewhere. However, these results pointed to the need for a more careful experiment, in which extraneous mechanisms which could affect the swirl were reduced to a minimum. In particular, it seemed important to make the horizontal scale of the tank much larger than that of the stirring motions.

In § 2 of this paper various simple mathematical models of the stirring process are examined. These suggest that a basic anisotropy is necessary if the mean motion is to differ from solid body rotation, but that given suitable anisotropy a vortex can indeed be set up. They also suggest how the tendency to form a vortex may be quantified in terms of a diffusion coefficient for angular momentum. This coefficient may reasonably be compared to the diffusivity of material particles in the random walk caused by the stirring, to yield an 'efficiency' for the vortex generating process. If the mixing length for angular momentum were the same as for material particles the efficiency would be unity. In §§ 3, 4 we report an experiment designed to measure the efficiency of a particular method of stirring, by means of an oscillating rigid grid. To within the sensitivity of our method we could detect no tendency for the mean motion to depart from solid body rotation, i.e. the efficiency was not significantly different from zero ($< 5\,\%$ with $95\,\%$ probability).

Much of the theoretical discussion in this paper has probably appeared, in embryo at least, elsewhere in the literature. The present objective is to collect together some of the arguments which can be put forward for and against vortex generation by random stirring, so that their strengths and weaknesses can be compared. Innumerable variations are possible, so the selection is a personal one. The treatment given is only a sketch and rigour is not attempted.

2. Theoretical considerations

Cylindrical polar co-ordinates (r, θ, z) are taken in an inertial frame of reference. The corresponding components of fluid velocity are $(u', \bar{v}+v', w')$, where

$$\overline{u'} = \overline{v'} = \overline{w'} = 0.$$

Averages are taken over a time long compared to the scale of the fluctuations (u', v', w') induced by the stirring, but short compared to that for the development of substantial changes in the mean swirl $\bar{v}(r, z, t)$. In the situation envisaged, we have initially

$$\bar{v} = \Omega r$$

corresponding to rigid body rotation about the vertical axis. The length scale L of the fluctuations is substantially smaller than the scale r of the mean motion. The stirring is presumed strong, i.e.

$$\Omega L \ll u' \tag{3}$$

and to a first approximation independent of time and homogeneous in the horizontal plane. It is induced by random body forces F. We shall consider in turn a number of special cases. To begin with, the influence of solid boundaries and variations of mean quantities in the vertical will be ignored.

(a) *Axisymmetric motion in an inviscid fluid*

When stirring is random but axisymmetric (independent of θ, and $F_\theta = 0$), it seems very probable that mean vortex can be induced. In an inviscid fluid, the angular momentum per unit mass of a ring moving radially

$$\gamma = (\bar{v}+v')r$$

is then completely conserved, so that

$$\frac{d\gamma}{dt} = 0. \tag{4}$$

If at time $t = 0$ the whole fluid is rotating as a solid body, we know that

$$\gamma_0 = \Omega r_0^2.$$

Subsequently the value of γ at any point (r, z) is determined by the original radius r_0 of the fluid particle present there. Even after many time-scales for the fluctuations, because $L/r \ll 1$, the change in radius is almost everywhere small, i.e.

$$r - r_0 \ll r_0,$$

and

$$\gamma \sim \bar{\gamma}_0 - \frac{d\bar{\gamma}_0}{dr}(r - r_0)$$

$$\sim \Omega r_0^2 - 2\Omega r_0(r - r_0). \tag{5}$$

The outward mean flux of angular momentum crossing unit area of cylindrical surface of radius r is

$$\overline{\gamma u'} \sim \bar{\gamma}_0 \overline{u'} - \frac{d\bar{\gamma}_0}{dr}\overline{u'(r - r_0)}.$$

Now $\overline{u'}$ vanishes, and

$$\overline{u'(r - r_0)} = \frac{1}{2}\frac{d}{dt}\overline{(r - r_0)^2}.$$

Under the assumptions made here this is the rate of increase of the one-dimensional dispersion $\frac{1}{2}\overline{(r - r_0)^2}$ of a cloud of marked fluid particles originally at (r_0, z) but subject to different realizations of the velocity fluctuations. It is also equal to the horizontal diffusion coefficient κ for a single small Lagrangian marker in the fluid, and is directly measurable. If genuine mixing is occurring it is definitely greater than zero. Thus there is a mean radial flux per unit area of angular momentum per unit mass equal to

$$M = -\kappa \frac{d\bar{\gamma}_0}{dr}. \tag{6}$$

(b) *Axisymmetric motion with internal friction*

In the inviscid model considered in the previous paragraph the magnitude of the velocity fluctuations v' at a point would continue to increase indefinitely as the mixing progressed. To achieve a quasi-steady statistical state in which $\bar{v}$ changes slowly as the inward flux of angular momentum builds up a vortex, it is necessary to allow some interchange of angular momentum between a ring and its surroundings. This would inevitably be achieved by viscosity. The simplest assumption to illustrate the effect is to take

$$\frac{d\gamma}{dt} = -\lambda(\gamma - \bar{\gamma}) \tag{7}$$

so that a ring is continually losing at a constant rate the excess of its angular momentum above the mean value of its instantaneous surroundings. The total derivative d/dt is construed as following a fluid particle. If λ^{-1} is comparable with the fluctuation timescale, it is permissible to ignore in $\overline{d\gamma/dt}$ the slow change $\partial\bar{\gamma}/\partial t$ compared with the effect $\overline{u'(\partial\gamma/\partial r)}$ of changes in position, and we have

$$\frac{d}{dt}(\gamma - \bar{\gamma}) + \lambda(\gamma - \bar{\gamma}) = -\frac{\partial\bar{\gamma}}{\partial t}u'.$$

This equation may be solved to give

$$\gamma(r,z,t) = \overline{\gamma}(r,z) - \frac{\partial \overline{\gamma}}{\partial r}\int_0^\infty u'(t-s)\, e^{-\lambda s} ds \tag{8}$$

where the integral is taken following a given particle. This shows that the instantaneous value of the angular momentum at a point depends predominantly on the more recent history of radial motions of the fluid particle which is instantaneously there, and departures from the local quasi-steady mean are fairly small. There is still, however, a mean radial flux of angular momentum of magnitude

$$M = -\frac{\partial \overline{\gamma}}{\partial r}\int_0^\infty S(s)\, e^{-\lambda s} ds \tag{9}$$

where $$S(s) = \overline{u'(t)\, u'(t-s)}$$

is the Lagrangian autocorrelation function for radial velocity. If λ is large, the integral $\sim \overline{u'^2}/\lambda$, whereas if λ is small it $\sim \kappa$, the particle diffusion coefficient introduced before. On any reasonable model, the integral is always positive and a significant fraction of κ. There is then a substantial down-gradient flux of angular momentum, described by a diffusion coefficient k comparable to κ and approximately independent of position.

(*c*) *Mixing by expanding and contracting fluid disks*

The axisymmetric motions considered above are very specialized, but other variations of the same general idea are possible. In one of these, attention is concentrated on a small disk-shaped fluid element lying in the horizontal plane anywhere in the rotating mass. The disk of fluid will tend to rotate about a vertical axis with an angular velocity of the same sign as Ω. If now it is distorted by the stirring so that it expands in the horizontal plane, the vertical vorticity decreases, the disk rotates more slowly, and the fluid particles more remote from the axis of the whole mass have a smaller swirl velocity than previously, whereas those particles which have moved nearer the axis swirl faster. On mixing with their new surroundings, the former particles tend to slow these down, the latter to speed them up. The result is an inward radial transfer of angular momentum. A fluid element which contracts in the horizontal plane, on the other hand, obtains an increased vertical vorticity, but the transfer of angular momentum has the same sign.

It should be noticed that this argument assumes that during the expansion or contraction phase there is on the average no net pressure force $(\overline{\partial p'/\partial\theta})$ on the element tending to alter its angular momentum as a whole. This possibility can be rigorously eliminated only for elements which are complete rings.

(*d*) *Non-axisymmetric stirring*

A simple model of a mixing process is one in which particles are repeatedly released with randomly distributed velocities, travel with those velocities for a short time τ, and are then mixed again with their new surroundings. Such a model is really more appropriate for a gas consisting of discrete molecules, rather than a liquid in which pressure forces act continuously, but it is similar to mixing length theories of turbulence. The result is included here because it illustrates cogently how the radial flux of angular momentum depends on the anisotropy of the mixing, and that when it is isotropic the flux does indeed vanish in a state of solid body rotation, rather than when the mean angular momentum is uniform.

The central statistical assumption is that, if the components of velocity in cylindrical co-ordinates are $(u', \bar{v}(r)+v', w')$, then the probability distribution $\nu(u', v', w', r)$ for a particle at the time of release is an even function of u', v', w', so that all odd moments like $\overline{u'}, \overline{v'^3}, \overline{u'v'}$ vanish. It is also necessary that the number of particles per unit volume be adjusted so that there is no radial flux of mass. It is then fairly straightforward to show that there is an average radial flux of angular momentum per particle

$$M = -\tfrac{1}{2}\tau\left\{\overline{u'^2}\frac{\partial}{\partial r}(r\bar{v}) - 2\overline{v'^2}\bar{v}\right\} + O(\tau^2). \tag{10}$$

If $\overline{v'^2} = 0$, this reduces to the expression for axisymmetric stirring obtained in § 2 (*a*). If $\overline{u'^2} = \overline{v'^2}$, on the other hand, M vanishes if and only if $r^2(\partial/\partial r)(\bar{v}/r) = 0$, i.e. when there is solid body rotation. Even though the angular momentum per particle is constant during the motion, and the total is conserved upon mixing, the equilibrium state is *not* one of uniform angular momentum. This arises because the components of velocity of a particle travelling in a straight line vary with time t, being, to first order in t,

$$\{u' + (\bar{v}+v')^2 t/r,\ (\bar{v}+v')(1-u't/r),\ w'\},$$

where u', v', w' refer to the moment of release $t = 0$. The second component shows that the instantaneous angular momentum

$$\gamma = (\bar{v}+v')(1-u't/r)(r+u't) + O(t^2)$$

is conserved, whereas the first describes an apparent radial acceleration proportional to $(\bar{v}+v')^2$. Thus, even though at the moment of release as many particles with given v' may be moving radially inwards as outwards ($\overline{u'v'} = 0$), it does not remain so, because there is a systematic bias against the particles with large γ moving radially inwards. This is a kinematic consequence of the curvature of the co-ordinate system. The bias counteracts the flux of angular momentum due to inward moving particles having originated in a region of larger $\bar{\gamma}$. If $\overline{v'^2} = \overline{u'^2}$ then in a state of solid body rotation the effects precisely cancel.

Although this model is oversimplified, it can easily be generalized to include mean radial pressure gradients, provided these act on particles in an unbiased manner. It also can describe a continuous liquid, provided the change in angular momentum of a fluid element due to the pressure forces acting on it can be modelled by a sequence of random step functions at regular intervals. Although this is certainly an inadequate picture of turbulent mixing, the model does show that the first argument of the introduction is deceptively simple and misleading.

(e) A linear, isotropic fluid

Before we discuss the explicit model which is to be tested experimentally, we will consider another argument which shows that anisotropy of the mixing is important if a vortex is to be produced. Batchelor (1967) points out that if the stress tensor M_{ij} describing the mean transfer of angular momentum in a fluid is a linear function of the local mean velocity gradients only,

$$M_{ij} = a_{ijkl}\frac{\partial \overline{u}_k}{\partial x_l} \tag{11}$$

and if the tensor a_{ijkl} describing the response of the fluid is isotropic, then we have by a well-known result

$$a_{ijkl} = a\,\delta_{ij}\,\delta_{kl} + b\,\delta_{ik}\,\delta_{jl} + c\,\delta_{il}\,\delta_{jk}. \tag{12}$$

Since, in the absence of couple stress, M_{ij} is symmetric, we must have $b = c$ and M_{ij} can only depend on the symmetric part of $\partial \overline{u}_k/\partial x_l$ and not on the antisymmetric part which describes the rotation. Thus, in a state of solid body rotation, M_{ij} should vanish. This argument is fundamental to the introduction of Newtonian viscosity in the Navier–Stokes equations, but should be applicable also on a larger scale. The key assumptions from the point of view of this paper are that the response of the stirred fluid to gradients of mean motion is both linear and isotropic. These are not *a priori* self evident, and although this argument imposes severe limitations on the type of mechanism which could produce a vortex, the existence of such a mechanism cannot be ruled out.

Anisotropic mixing does appear to be vital in a completely different model treated by Kippenhahn (1963). He has considered convective motions in a spherical shell of gas, and concludes that differential rotation between the poles and the equator is only possible if the small-scale mixing is anisotropic.

(f) The influence of the Coriolis restoring force

Finally, we will consider the second argument mentioned in the introduction. A particle in a uniformly rotating fluid displaced radially through a small distance σ experiences in general a restoring force of magnitude comparable with $4\Omega^2$ times the displacement. If such a fluid is mixed, it was argued, work is done by the stirring forces against this restoring force, and this work must reappear as kinetic energy of the mean motion. This statement cannot be strictly valid, for in the absence of dissipative forces the fluctuations in velocity would go on increasing indefinitely. With dissipation, some at least of this kinetic energy is lost during the mixing of a fluid element with its surroundings, as we saw in the axisymmetric case described above. This mixing is essential for the radial transfer of angular momentum. Without detailed consideration of the whole energy budget it seems impossible to estimate how much of the input will reappear as an increase of the mean swirl. However, the restoring force invoked is a quadratic function of the rotation rate, and it is possible that it influences the turbulent structure in such a way that some anisotropy is introduced, evading the negative conclusion of §2(*e*). Thus a radial transfer of angular momentum cannot be ruled out.

(g) Theoretical conclusions

In the opinion of the present authors the theoretical discussion given here is inconclusive. The first argument described in the introduction (and Prandtl's reasoning) is fallacious, the second argument is not rigorous. The possibility of forced stirring causing the formation of a vortex in an initially uniformly rotating fluid cannot be ruled out, but neither has it been shown that a method of stirring which is not preferentially orientated relative to the local radial direction can lead to a vortex. Hence an empirical test is required, though it must be remembered that a single experiment can only be conclusive if it yields a positive result. If (as here) no vortex is observed, it could be that the method of stirring is inappropriate. Further experiments are desirable.

3. Experimental considerations

(a) The working hypothesis

In order to measure the strength of the tendency towards vortex formation, it is essential to have a theoretical framework within which the results can be analysed. For a fluid in which pressure forces are continuously acting on particles, it is not obvious that the model of 2(*d*) is relevant. Nevertheless, the expression (10) contains the suggestion that under different circumstances the radial angular momentum flux can be proportional to $\partial\overline{\gamma}/\partial r$ or to $r^2(\partial/\partial r)(\overline{v}/r)$. It appears reasonable to take as our hypothesis, which may be tested experimentally, a momentum flux per unit mass which includes both these terms in the form

$$M = -\kappa\left\{\epsilon\frac{\partial}{\partial r}(\overline{v}r)+\delta r^2\frac{\partial}{\partial r}\left(\frac{\overline{v}}{r}\right)\right\}, \tag{13}$$

where ϵ and δ are positive constants of order unity or less. Although two terms are introduced here, it will be shown that the conditions in our experiments are such that the second is negligible, and no information is obtained about the magnitude of $\delta\kappa$. The coefficient of the first term, $k = \epsilon\kappa$, on the other hand, is a diffusion coefficient for angular momentum, the magnitude of which may be assessed from the experiments. ϵ measures the ratio of the effectiveness of the mixing in transferring angular momentum to that for the transfer of fluid particles, and for want of a better name is called here the 'efficiency factor'.

At this stage we should point out that so far no mention has been made of variations in $\overline{v}$ or of the fluctuation statistics with distance z along the axis of rotation, nor has the presence of bounding surfaces been allowed for. In our experiment a solid grid was rotating with constant angular velocity Ω and there was a rapid transfer of relative angular momentum $(\overline{v}-\Omega r)r$ to the grid and container. This of course varied markedly with z, but for the small relative velocities envisaged here it seems reasonable to assume that the vertically averaged values of $\overline{v}$ obey a linear decay law

$$\frac{\partial\overline{v}}{\partial t} = -\lambda(\overline{v}-\Omega r), \tag{14}$$

where λ is a constant determined by the turbulence. The decay time λ^{-1} was measured; it is comparable with the fluctuation time scale and is very much shorter than the time for a mean relative swirl due to a radial diffusion of angular momentum to grow in the absence of this transfer.

The final form of the hypothesis made here is that an expression like equation (13) holds also for the vertically averaged values of M, κ and $\bar{v}$, even in the presence of the grid and horizontal bounding surfaces. Consideration of the angular momentum balance for a ring of radius Γ then shows that

$$\frac{\partial \bar{v}}{\partial t} = \frac{\epsilon\kappa}{r^2}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}(r\bar{v})\right) + \frac{\delta\kappa}{r^2}\frac{\partial}{\partial r}\left(r^3\frac{\partial}{\partial r}\left(\frac{\bar{v}}{r}\right)\right) - \lambda(\bar{v}-\Omega r).$$

This may be re-written in terms of the relative velocity $\bar{v}-\Omega r$ as

$$\frac{\partial}{\partial t}(\bar{v}-\Omega r) = \frac{\epsilon\kappa}{r^2}\frac{\partial}{\partial r}\left\{r\frac{\partial}{\partial r}(r(\bar{v}-\Omega r))\right\} + \frac{\delta\kappa}{r^2}\frac{\partial}{\partial r}\left\{r^3\frac{\partial}{\partial r}\left(\frac{\bar{v}-\Omega r}{r}\right)\right\} - \lambda(\bar{v}-\Omega r) + 4\epsilon\kappa\Omega/r. \tag{15}$$

Now in our experiment $\lambda r^2/\kappa$ was very large, except very near the axis, and the relative velocities were very much smaller than the absolute ones. The first and second terms on the right-hand side of equation (15) are negligible compared to the third, and after a time of order λ^{-1} a steady state is reached in which there is a local balance between the radial diffusion of absolute angular momentum and its transfer to the container at a rate proportional to the relative velocity,

$$\bar{v}-\Omega r = 4\frac{\kappa\epsilon}{\lambda}\frac{\Omega}{r}. \tag{16}$$

Thus, except for a region of width of order L around the axis and any side walls to the container, the mean motion predicted by theory is a slow relative vortex, of magnitude directly proportional to the diffusion coefficient $k = \epsilon\kappa$ for angular momentum. The remainder of this paper is devoted to describing the experiments and results, which lead to the determination of an upper limit for the efficiency factor ϵ.

(b) *Design of the apparatus*

The preliminary observations suggested that it was desirable to stir homogeneously on a scale substantially smaller than the radius of the container, and to reduce the mean meriodinal circulation to a minimum. A steady-state experiment, rather than a transient one, also seemed preferable. With these criteria in mind, the experimental tank chosen was a steel drum 71 cm in diameter, filled to a depth of 12 cm with liquid having a free surface. Kerosene was used throughout as the working fluid (for historical reasons which need not concern us here). This was stirred mechanically with a grid of plastic strips 1·2 cm thick and 1·0 cm wide, arranged in a horizontal plane with spacing 5·0 cm. The grid was supported on thin rods from a frame above it, and oscillated vertically in simple harmonic motion through the middle of the liquid layer (see figure 1). This design, with the grid itself and its support made as rigid as possible, was decided on after preliminary runs had revealed a spurious circulation attributable to the 'pumping action' caused by the flexing of a thinner grid.

The whole apparatus was mounted on the rotating table previously built at Woods Hole by Dr A. J. Faller. This turntable permits accurately controlled rotation up to 15 r.p.m. and can be viewed by a synchronously rotating television camera mounted vertically above it. Measurements of the displacements on the

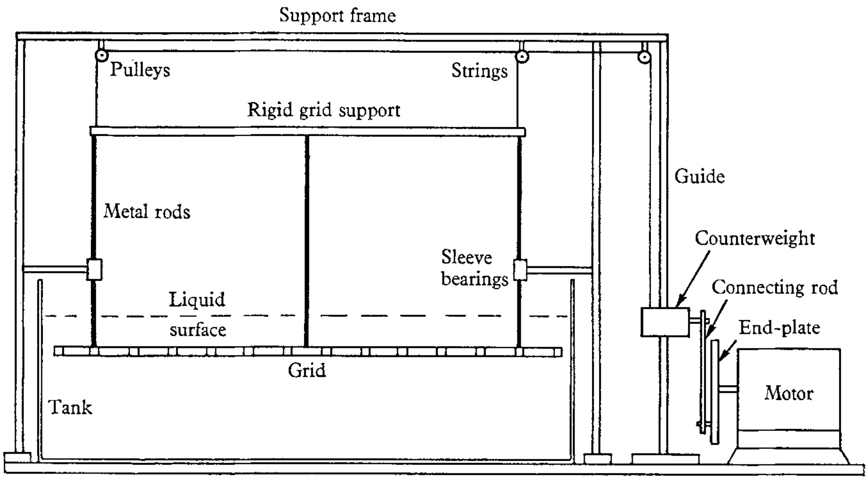

FIGURE 1. Schematic elevation of the experimental tank, showing the mechanism used to oscillate the grid.

television screen provide a sensitive indicator of velocities relative to the rotating tank. There was no detectable variation of the properties of the turbulence with rotation rate Ω, although the inverse Rossby number $\Omega L/U$ based on the grid spacing L ($= 5$ cm) and the r.m.s. grid velocity U (typically 10 cm/sec) varied between zero and 0·5.

A consequence of the experimental method adopted is the large damping of any mean swirl relative to the grid, because of the turbulent transfer of angular momentum to the grid. When the grid was stationary the decay time of such large-scale swirl was of the order of minutes, but while the grid was being oscillated it was a few seconds. However, the lack of control and probable meridional circulation associated with stirring by other methods which were considered, such as thermal convection or the injection of air bubbles, seemed far more serious.

(c) *The technique of velocity measurement*

It is clear from equation (16) that several measures of fluid velocity are required. All of these were obtained by tracking a small neutrally buoyant marker particle, a plastic sphere about 6 mm in diameter, which was free to move vertically through the grid bars, sampling the whole depth as well as the area of the tank. Provided long enough times are considered for the particle to visit every region in the container, the time mean of any quantity following a fluid particle is equal to the volume mean of the same quantity.

For the measurement of angular displacement an annular region was marked out on the television screen, corresponding to radii of 7·3 cm and 22·0 cm in the fluid. The centre and edge of the container were excluded because (16) breaks down there, and because of the difficulty of defining angular displacements of particles moving across the centre. The angular co-ordinate of the marker particle was recorded (generally to within $\pm 10°$) for each 3 sec interval at the beginning of which the image was within the annulus. With the velocity distribution described by (16), the volume average of the mean angular velocity in an annular region between radii c and d is

$$\bar{\dot{\phi}} = \frac{4k}{\lambda}\Omega \ln (d/c)/\tfrac{1}{2}(d^2-c^2) \tag{17}$$

relative to the rotating frame. The periods over which this was taken (of order of an hour) were such that the area sampling was very satisfactory; the observed times spent inside and outside the annulus (see table 1) corresponded closely to the known volume ratio (35 % inside the annulus).

The decay constant λ was too large to be estimated by observing directly the exponential decay to zero of a swirl. Instead, particles were observed during a period when the rotation rate Ω of the tank was being altered at a constant rate $\dot{\Omega}$, through a total range $[\Omega]$. During this process, the solution of equation (14) gives

$$\bar{v} - \Omega r = -\frac{\dot{\Omega}}{\lambda} r.$$

Thus the mean angular displacement during the spin-up process should be independent of radius and equal to

$$[\bar{\phi}] = -\frac{1}{\lambda}[\Omega]. \tag{18}$$

This equation will of course only be valid if $\dot{\Omega}/[\Omega]\lambda \ll 1$, which was true in our experiments.

For the measurement of horizontal mixing, the television screen was divided into 400 numbered squares, each corresponding to 3·7 cm square in the fluid. At 3 sec intervals the square containing the particle was noted. From sets of twenty random walks the mean square displacement $\overline{r^2}$ was computed, r being the displacement, after time t, of a particle from its initial position. The equation

$$\overline{r^2} = 4\kappa t \tag{19}$$

then gave a value for κ.

Our final result will be stated as an upper limit on the ratio k/κ which has been defined as the efficiency ϵ of the mixing process for angular momentum. Combining (17), (18) and (19) we have

$$\epsilon = \frac{k}{\kappa} = \left(\frac{\bar{\dot{\phi}}}{\Omega}\right)\frac{[\Omega]}{[\bar{\phi}]}\left(\frac{t}{\overline{r^2}}\right)\frac{\tfrac{1}{2}(d^2-c^2)}{\ln (d/c)}. \tag{20}$$

All the terms in this expression have been measured directly; the detailed results will be given in the next section.

4. Experimental results

(a) Relative angular velocity

The stirring rate was held near 1·3 c/s and the vertical excursion of the grid at 5 cm for this and all the other quantitative experiments. The measurements of the mean relative angular motion in the annulus between $c = 7{\cdot}3$ cm and $d = 22{\cdot}0$ cm were conducted at a basic rotation rate of 12 r.p.m. The total rotation in each of four runs was small, as shown in table 1, not significantly different from zero or from that in two control runs with the tank at rest. The significance was assessed by regarding the relative rotation in each 3 sec interval as an independent measure of the angular motion, and evaluating the variance of the deviations from zero. The total number of observations and the sum of the squares of the deviations are shown in the table. The final result for the four runs at 12 r.p.m. taken together is $\dot{\phi}/\Omega = 0{\cdot}0007 \pm 0{\cdot}0015$, the limit quoted being the standard error of the mean.

Experiment no.	Rotation rate Ω (r.p.m.)	Number of 3 sec intervals in annulus	Total relative rotation (20° units)	Sum of squares of deviations (20° units)^2	Proportion of total time in annulus (%)
1	12	344	−5·0	169	41
2	12	404	+1·0	166	30
3	12	412	−17·5	130	—
4	12	498	+8·5	209	41
5	0	263	+5·5	133	32
6	0	390	+13·0	183	37

TABLE 1. The measurements of relative angular motion, made by following a particle over many 3 sec intervals, at the beginning of which the particle was within the annulus

Also shown in the last column of table 1 is the proportion of the total time which the marker particle spent in the annulus. This information was used to assess how well the particle sampled the volume of the tank.

(b) The decay rate

In accordance with equation (18) the angular motion of a particle during a period of acceleration is required. In practice, an average over 30 runs was obtained, during which Ω was changed with constant acceleration through the range 0–12 r.p.m. or back again. The spin-up time was restricted to be about 30 sec by the design characteristics of the rotating table. Only those runs were included for which the particle both began and ended in the annulus used in the relative velocity measurements. The mean value obtained in this way is

$$\lambda = -[\Omega]/[\overline{\phi}] = 0{\cdot}74 \pm 0{\cdot}07\,\mathrm{s}^{-1}.$$

This method implies that we can assume that λ is independent of the rotation rate. For the range of stirring Rossby numbers used this is probably a good first approximation. The other measured parameter of the turbulence κ does seem to be independent of Ω within the error of observation.

(c) *Diffusion coefficient for particles*

The mean-square deviations of particles from their initial position near the centre was measured in four separate runs, two without rotation and one each at 6 r.p.m. and 12 r.p.m. During each run, averages of r^2 were computed over 20 particles at 3 sec intervals, taking our zero 3 sec from the beginning of stirring so that the turbulence would be properly established through the whole run. The

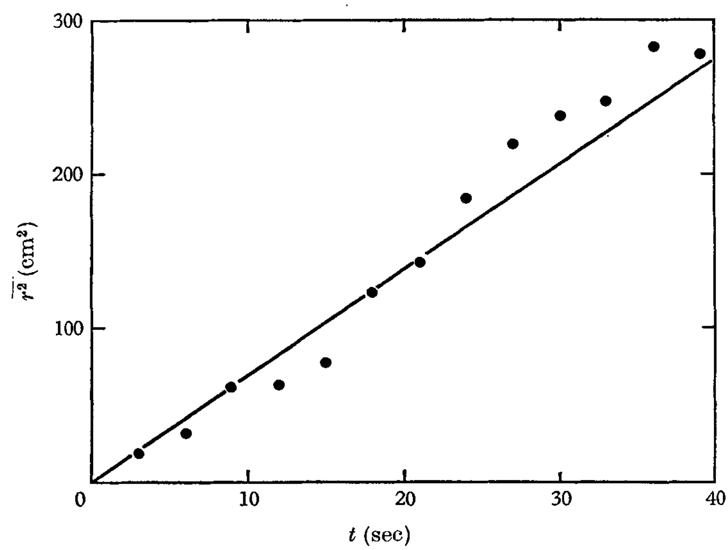

FIGURE 2. A typical plot of mean dispersion as a function of time (run B in table 2). The line drawn gives equal weight to the slopes determined from the individual points.

Experiment no.	Rotation rate (r.p.m.)	$\overline{r^2}/t = 4\kappa$ cm^2 s^{-1}	S.E. of mean
A	0	7·8	1·0
B	0	6·8	0·8
C	6	7·2	1·1
D	12	6·6	0·5

TABLE 2. The measurements of dispersion of fluid particles. Each run is an average over 20 observations of a marker particle started near the centre of the tank

mean-square displacement was proportional to time up to about 40 sec, as shown in figure 2 for a typical run. After this it increased more slowly, probably because a significant proportion of the particles was affected by the outer boundary of the tank at $r = a = 35$ cm. For very long runs, $\overline{r^2}$ should be constant and equal to $\frac{1}{2}a^2$.

The values of $4\kappa = \overline{r^2}/t$ obtained using the linear range of the four experiments are shown in table 2. The limits quoted are standard errors of the mean, computed by supposing that the whole history of each particle defines an independent estimate of $\overline{r^2}/t$ and giving equal weight to the dispersion at each of the recorded times. The values are not significantly different at the different rotation rates, so we can lump all the results together and use the mean value

$$4\kappa = \overline{r^2}/t = 7{\cdot}1 \pm 0{\cdot}5\,\mathrm{cm^2\,s^{-1}}.$$

(*d*) *Calculation of the efficiency*

Finally, we use (25) to calculate the efficiency for mixing of angular momentum. The numerical values obtained above, and the standard errors of the means, are grouped together for convenience in table 3. The final value is

$$\epsilon = 0{\cdot}014 \pm 0{\cdot}031; \tag{21}$$

note that this standard error depends on the standard error of $\dot{\phi}/\Omega$ and on the mean values of the other factors, but hardly at all on the other standard errors. Assuming a Gaussian distribution of $\dot{\phi}/\Omega$ we may state (21) in alternative ways according to the significance level adopted. A convenient interpretation is the following: *the transfer coefficient for angular momentum in our experiment is not significantly different from zero, and is less than* 5 % *of that for fluid particles, with probability* 95 %.

Quantity	$\dot{\phi}/\Omega$	λ ($\mathrm{s^{-1}}$)	4κ ($\mathrm{cm^2\,s^{-1}}$)	$\dfrac{\frac{1}{2}(d^2-c^2)}{\ln(d/c)}$ ($\mathrm{cm^2}$)
Mean	0·0007	0·74	7·1	196
Standard error	± 0·0015	± 0·07	± 0·5	± 10 (estimated)

TABLE 3. Summary of our experimental results

5. Final remarks

The experiment described preceded in point of time much of the theoretical analysis, and although it seems to provide a satisfactory test of the rather vague suggestions described in the introduction, it is clear in retrospect that there are several modifications of the experimental techniques which it would be useful to try in the future. It would clearly be desirable to reduce the resistance of the grid, and its damping effect on the swirl. This presented difficulties when the aim was to produce homogeneous stirring, isotropic in the horizontal, but it is less of a problem when one considers anisotropic stirring, which is suggested by § 2 (*e*). An array of concentric rings with equal radial spacing would be a suitable geometry both to introduce radial-circumferential anisotropy and reduce the resistance to swirl. A positive effect of this stirring would then be expected.

Another mechanism for the production of a relative angular motion by stirring a rotating vessel, different from that considered here, has been proposed by Gough & Lynden-Bell (1968). They present experimental evidence for such a swirl produced in a thin layer of fluid by convective stirring, and also discuss why their mechanism should not be effective for our case of mechanical stirring. The essential point is that the time needed for vorticity to diffuse to the boundary, in their model, is just not available when the field of motion is being rapidly changed by external stirring. With this mechanism in mind, an experiment where the fluid is stirred uniformly by one double pass of a grid, which is then removed from the fluid, again becomes an attractive possibility.

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